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Branch conditions for wave dislocations in light beams

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Abstract. The densities of wave dislocations (or topological charge densities of wave dislocations) in terms of the complex scalar wave are given from the definition of topological charges of wave dislocations. The branch conditions for generating, annihilating, colliding, splitting and merging of wave dislocations are obtained according to the properties of the complex scalar wave itself. It is found that the velocities of wave dislocations are infinite when they are annihilating or generating, which is obtained only from the topological properties of the complex scalar wave.

1. Introduction

Wave dislocations (phase singularities) in optical fields have drawn great interest because they are of importance for understanding fundamental physics and have many important applications. The generation, annihilation, collision, split and mergence of wave dislocations have been studied intensively in many ways [1–20]. Berry, Nye and collaborators [9–20] considered the scalar wave equation and proved the existence of wave dislocations by exhibiting a number of special solutions of the scalar wave equation that have the dislocation properties, which showed dislocations colliding, annihilating, generating, and so on. In this paper, we will study the conditions for generating, annihilating, colliding, splitting and merging of wave dislocations by making use of the ϕ-mapping topological current theory which plays an important role in discussing the topological invariants and structure of physical systems [21–23] and has been used to study the mathematical framework for wave dislocations in three-dimensional space [24]. A useful condition for branch processes (generation, annihilation, collision, split and mergence of wave dislocations) will be given, from which one can find the positions of the branch points in light beams, that is, the branch points must be subject to the constraint condition that the usual Jacobian $D(\phi/x)$ vanishes. Then, according to the values of the vector Jacobians of the complex scalar wave, the branch points are classified into two types: limit points and bifurcation points; wave dislocations generate or annihilate at the limit points and collide, split or merge at the bifurcation points of the complex scalar wave.

This paper is organized as follows: in section 2, the densities of wave dislocations in terms of the complex scalar wave are given by means of the ϕ-mapping topological current theory. The velocity field of wave dislocations in light beams is also obtained. In section 3, from the topological properties of the complex scalar wave, the conditions for generating, annihilating, colliding, splitting and merging of wave dislocations are obtained and several crucial cases of

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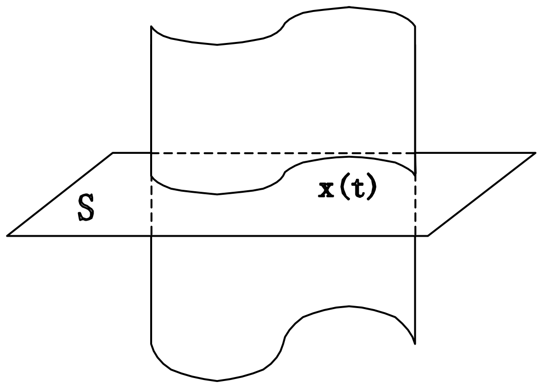

Figure 1. S is one cross section normal to the z-axis. $X(t)$ is the intersection curve between the evolution surface of a wave dislocation and the cross section S, i.e. the movement curve of the wave dislocation on the cross section S.

branch process are discussed in detail. In section 4, we show several examples in the literature that explicitly satisfy the branch conditions. We would like to emphasize, however, that the conditions for branch processes are universal and all the branch processes must be subject to them. We give our concluding remarks in section 5.

2. Densities of wave dislocations in light beams

Wave dislocations are topological objects and possess topological charges, which can be attributed to the helicoidal spatial structure of the wavefront around a phase singularity. This structure is similar to a crystal lattice defect, and therefore was first known as a wave dislocation [9]. There are three types of dislocations: edge, screw and mixed edge–screw dislocations. An edge dislocation is located along a line in the transverse plane and travels with the wave. A screw dislocation is called an optical vortex, the essential property of which is that the phase changes by $2\pi m$ on a closed circuit around it. m is called the topological charge, which is positive for a right-screw helicoid ($m > 0$), and vice versa. At the locations of these three kinds of dislocations the wave amplitude becomes zero and the phase is indeterminate [25, 26].

Let there be a complex scalar wave $\psi(\vec{r}, t)$ propagating along the z-axis in (3 + 1)-dimensional space-time, in which there exist evolution surfaces formed by the movements of the wave dislocations. For simplicity, let us take an arbitrary cross section normal to the z-axis, i.e. (2 + 1)-dimensional space-time with coordinates $x^1 = x$, $x^2 = y$ and $x^0 = t$. The intersection lines between the evolution surfaces and the cross section are just the motion curves of the wave dislocations on the cross section (see figure 1). One thing to point out is that if one takes the cross section everywhere along the z-axis, the motion properties of wave dislocations will be given completely.

The complex scalar wave $\psi(\vec{\rho}, t)$ is denoted as $\psi(\vec{\rho}, t) = \phi^1(\vec{\rho}, t) + \mathrm{i}\phi^2(\vec{\rho}, t)$, where $\phi^1(\vec{\rho}, t)$ and $\phi^2(\vec{\rho}, t)$ can be denoted as two components of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$. The zero points of the complex scalar wave $\psi(\vec{\rho}, t)$, i.e.,

$$\phi^1(x^1, x^2, t) = 0 \qquad \phi^2(x^1, x^2, t) = 0 \tag{1}$$

determine the positions of wave dislocations on the cross section. If the Jacobian determinant $D(\phi/x) \neq 0$, the solutions of equations (1) are generally expressed as

$$x^1 = x_l^1(t) \qquad x^2 = x_l^2(t) \qquad l = 1, 2, \ldots, N \tag{2}$$

which represent the motion curves of N wave dislocations $\vec{\rho}_l (l = 1, 2, \ldots, N)$ on the cross section, and which show the wave dislocations moving in (2 + 1)-dimensional space-time.

The topological charge of the lth wave dislocation (or the generalized winding number W_l of $\vec{\phi}$ at the lth zero point $\vec{\rho}_l$) is defined by the Gauss map n: $\partial\Sigma_l \longrightarrow S^1$ [21]:

$$W_l = \frac{1}{2\pi}\int_{\partial\Sigma_l} n^*(\epsilon_{ab}n^a\,\mathrm{d}n^b) \qquad n^a = \phi^a/\|\phi\| \tag{3}$$

where n^* is the pullback of Gauss map n, $\partial\Sigma_l$ is the boundary of a neighbourhood Σ_l of $\vec{\rho}_l$ and $\Sigma_l \cap \Sigma_m = \emptyset$ for Σ_m is the neighbourhood of another arbitrary wave dislocation $\vec{\rho}_m$. In topology it means that, when the point $\vec{\rho}$ covers $\partial\Sigma_l$ once, the unit vector $\vec{n}$ will cover S^1, or $\vec{\phi}$ covers the corresponding region W_l times, which is a topological invariant. Using Stokes' theorem in the exterior differential form, one can deduce that

$$W_l = \frac{1}{2\pi}\int_{\Sigma_l} \epsilon_{ab}\epsilon^{jk}\partial_j n^a \partial_k n^b\,\mathrm{d}^2x \qquad j, k = 1, 2. \tag{4}$$

So, it is clear that the densities of wave dislocations (or the topological charges densities) are just

$$\rho = \frac{1}{2\pi}\epsilon_{ab}\epsilon^{jk}\partial_j n^a \partial_k n^b \tag{5}$$

which is the time component of the topological current of the two-dimensional vector field $\vec{\phi}$ [27]

$$J^i = \frac{1}{2\pi}\epsilon^{ijk}\epsilon_{ab}\partial_j n^a \partial_k n^b \qquad i = 0, 1, 2. \tag{6}$$

Obviously, the topological current is identically conserved,

$$\partial_i J^i = 0. \tag{7}$$

Following the ϕ-mapping topological current theory, it can be rigorously proved that

$$J^i = \delta^2(\vec{\phi})D^i\left(\frac{\phi}{x}\right) \tag{8}$$

where the Jacobians $D^i(\phi/x)$ are defined as $D^i(\phi/x) = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b$, in which $D^0(\phi/x)$ is the usual two-dimensional Jacobian $D(\phi/x)$. Now, the densities of wave dislocations are expressed in terms of the complex scalar wave $\psi(\vec{\rho}, t)$:

$$\rho = \delta^2(\vec{\phi})D\left(\frac{\phi}{x}\right). \tag{9}$$

Here, one can see that the densities of wave dislocations in terms of the complex scalar wave (9) are obtained directly from the definition of topological charges of wave dislocations (winding numbers of zero points of the complex scalar wave), which is useful because it avoids the problem of having to specify the position of the wave dislocations explicitly, and is more general than usually considered.

According to the δ-function theory and the ϕ-mapping topological current theory, one can prove that

$$\delta^2(\vec{\phi}) = \sum_{l=1}^{N}\frac{\beta_l}{|D(\phi/x)_{\vec{\rho}_l}|}\delta^2(\vec{\rho} - \vec{\rho}_l) \tag{10}$$

where the positive integer β_l is called the Hopf index [22] of map $x \longrightarrow \phi$. The meaning of β_l is that when the point $\vec{\rho}$ covers the neighbourhood Σ_l of the zero $\vec{\rho}_l$ once, the vector field $\vec{\phi}$ covers the corresponding region β_l times. By substituting equation (10) into (9), we obtain that

$$\rho = \sum_{l=1}^{N}\beta_l\eta_l\delta^2(\vec{\rho} - \vec{\rho}_l) \tag{11}$$

where η_l is the Brouwer degree [22]: $\eta_l = \mathrm{sgn}D(\phi/x)_{\vec{\rho}_l} = \pm 1$. One can find the relation between Hopf index β_l, Brouwer degree η_l and winding number W_l: $W_l = \beta_l \eta_l$ from equations (4) and (11). It is obvious that equation (11) represents N isolated wave dislocations of which the lth wave dislocation is charged with the topological charge $\beta_l \eta_l$. For the case of optical vortices (screw dislocations), the results coincide with [9, 26, 28], and have a more straightforward and strict significance. The lth optical vortex in light beams shows itself as a system of β_l helicoids, nested on the same axis, η_l is +1 for a counter-clockwise helicoid and -1 for a clockwise one.

The current densities of wave dislocations (N wave dislocations with the topological charges $\beta_l \eta_l$ moving in space) can be written in the same form as the current densities in hydrodynamics:

$$J^i = \sum_{l=1}^{N} \beta_l \eta_l \delta^2[\vec{\rho} - \vec{\rho}_l(t)] \frac{\mathrm{d}x_l^i}{\mathrm{d}t}. \tag{12}$$

According to equation (7), the topological charges of wave dislocations are conserved:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{13}$$

which is only the topological property of the complex scalar wave. Following the ϕ-mapping topological current theory, we can also get the velocity of the lth wave dislocation

$$\vec{\upsilon}_l = \frac{\mathrm{d}\vec{\rho}_l}{\mathrm{d}t} = \left.\frac{\vec{D}(\phi/x)}{D^0(\phi/x)}\right|_{\vec{\rho}_l} \qquad \vec{D}(\phi/x) = [D^1(\phi/x), D^2(\phi/x)]$$

from which one can identify the velocity field of wave dislocations as

$$\vec{\upsilon}(\vec{\rho}, t) = \frac{\vec{D}(\phi/x)}{D(\phi/x)} \tag{14}$$

where it is assumed that the velocity field is used inside expressions multiplied by the wave dislocations locating δ function. The expressions given by equation (14) for the velocity of wave dislocations are useful because they avoid the problem of having to specify the positions of the wave dislocations explicitly. The positions are implicitly determined by the zeros of the complex scalar wave. So, the location and the velocity of the lth wave dislocation are determined by the lth zero $\vec{\rho}_l(t)$ and the vector field $\vec{\upsilon}_l$ on $\vec{\rho}_l(t)$ respectively.

The solutions (2) of equations (1) are based on the condition that the Jacobian $D^0(\phi/x)|_{\vec{\rho}_l} \neq 0$. When $D^0(\phi/x)|_{\vec{\rho}_l} = 0$, i.e. η_l is indefinite, the above results (2) will change in some way. It is interesting to discuss what will happen and what the correspondence in physics is when this condition fails.

3. Branch conditions for generating, annihilating, colliding, splitting and merging of wave dislocations

When the usual Jacobian $D^0(\phi/x) = 0$ at some points along $\vec{\rho}_l$, it is shown that there exist several crucial cases of branch process at these points, which are called branch points. There are two kinds of branch points, namely limit points and bifurcation points. Each kind corresponds to different cases of branch processes. In the following will discuss them in detail.

First, we study the case that the zeros of the complex scalar wave ψ include some limit points. The limit points are determined by equations (1) and

$$D^0\left(\frac{\phi}{x}\right)\bigg|_{(\vec{\rho}_l, t)} = 0 \qquad D^1\left(\frac{\phi}{x}\right)\bigg|_{(\vec{\rho}_l, t)} \neq 0 \tag{15}$$

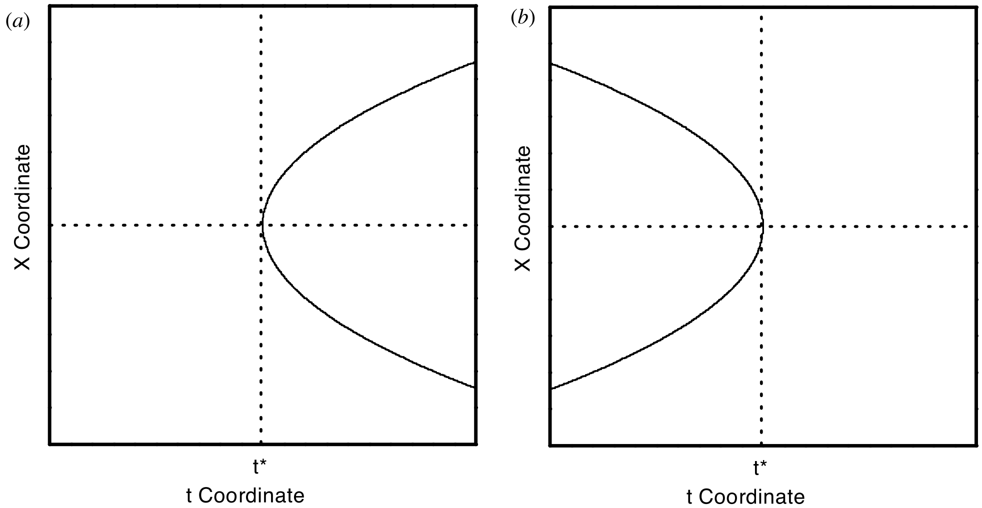

Figure 2. (*a*) The origin of two wave dislocations. (*b*) Two wave dislocations annihilate in collision at the limit point.

or

$$D^0\left(\frac{\phi}{x}\right)\bigg|_{(\vec{\rho}_l,t)} = 0 \qquad D^2\left(\frac{\phi}{x}\right)\bigg|_{(\vec{\rho}_l,t)} \neq 0. \tag{16}$$

For simplicity, we only consider case (15) and denote one of the limit points as $(\vec{\rho}_l, t^*)$. Taking account of (15) and using the implicit function theorem, we have a unique solution of equations (1) in the neighbourhood of the limit point $(\vec{\rho}_l, t^*)$:

$$t = t(x^1) \qquad x^2 = x^2(x^1) \tag{17}$$

with $t^* = t(x_l^1)$. From (15), it is easy to see

$$\frac{\mathrm{d}t}{\mathrm{d}x^1}\bigg|_{(\vec{\rho}_l,t^*)} = 0 \qquad \text{i.e.} \quad \frac{\mathrm{d}x^1}{\mathrm{d}t}\bigg|_{(\vec{\rho}_l,t^*)} = \infty. \tag{18}$$

Thus, the Taylor expansion of equation (17) in the neighbourhood of the limit point $(\vec{\rho}_l, t^*)$ is

$$t - t^* = \frac{1}{2}\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}\bigg|_{(\vec{\rho}_l,t^*)}(x^1 - x_l^1)^2 \tag{19}$$

which is a parabola in the $x^1 - t$ plane. From (19) we can obtain the branch solutions of wave dislocations at the limit point. If $\mathrm{d}^2 t/(\mathrm{d}x^1)^2|_{(\vec{\rho}_l,t^*)} > 0$, we have the branch solutions for $t \geqslant t^*$ (figure 2(*a*)), otherwise, we have the branch solutions for $t \leqslant t^*$ (figure 2(*b*)). The former is related to the origin of the wave dislocations, and the latter is related to the annihilation of the wave dislocations. One result of equation (18), that the velocity of wave dislocations is infinite when they are annihilating or generating, which is gained only from the topology of the complex scalar wave, agrees with that obtained by Nye and Berry [9].

Since the topological current is identically conserved, the topological charges of these two generated or annihilated wave dislocations must be opposite at the limit point, i.e. $\beta_1\eta_1 + \beta_2\eta_2 = 0$, which shows the generation and the annihilation of a pair of dislocations. For the case of optical vortices, it corresponds to the generation and annihilation of the vortex–antivortex pair.

Then, let us turn to the other case, in which the restrictions of equations (1) are

$$D^j\left(\frac{\phi}{x}\right)\bigg|_{(\vec{\rho}_l,t)} = 0 \qquad j = 0, 1, 2. \tag{20}$$

These three restrictive conditions will lead to an important fact that the functional relationship between t and x^1 or x^2 is not unique in the neighbourhood of $(\vec{\rho}_l, t^*)$. In our topological current theory, this fact is easily seen from

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = \frac{D^1(\phi/x)}{D^0(\phi/x)}\bigg|_{(\vec{\rho}_l,t^*)} \qquad \frac{\mathrm{d}x^2}{\mathrm{d}t} = \frac{D^2(\phi/x)}{D^0(\phi/x)}\bigg|_{(\vec{\rho}_l,t^*)} \tag{21}$$

which under (20) directly shows that the direction of the integral curve of (21) is indefinite at $(\vec{\rho}_l, t^*)$. Therefore, the very point $(\vec{\rho}_l, t^*)$ is called a bifurcation point of the complex scalar waves. With the aim of finding the different directions of all branch curves of equations (1) at the bifurcation point, we suppose that

$$\frac{\partial\phi^1}{\partial x^2}\bigg|_{(\vec{\rho}_l,t^*)} \neq 0. \tag{22}$$

And, according to the ϕ-mapping topological current theory, the Taylor expansion of the solution of equations (1) in the neighbourhood of the bifurcation point $(\vec{\rho}_l, t^*)$ can be expressed as [21]:

$$\alpha(x^1 - x_l^1)^2 + 2\beta(x^1 - x_l^1)(t - t^*) + \gamma(t - t^*)^2 = 0 \tag{23}$$

which leads to

$$\alpha\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2\beta\frac{\mathrm{d}x^1}{\mathrm{d}t} + \gamma = 0 \tag{24}$$

and

$$\gamma\left(\frac{\mathrm{d}t}{\mathrm{d}x^1}\right)^2 + 2\beta\frac{\mathrm{d}t}{\mathrm{d}x^1} + \alpha = 0 \tag{25}$$

where α, β and γ are three constants. The solutions of equations (24) or (25) give different directions of motion of the wave dislocations at the bifurcation point. There are four kinds of important situations.

First, $\alpha \neq 0$, $\Delta = 4(\beta^2 - \alpha\gamma) > 0$, from equation (24) we get two different solutions: $\mathrm{d}x^1/\mathrm{d}t\,|_{1,2} = (-\beta \pm \sqrt{\beta^2 - \alpha\gamma})/\alpha$ which is shown in figure 3, where two wave dislocations collide at the bifurcation point $(\vec{\rho}_l, t^*)$. This shows that two wave dislocations meet and then depart at the bifurcation point. Secondly, $\alpha \neq 0$, $\Delta = 4(\beta^2 - \alpha\gamma) = 0$, there is only one solution: $\mathrm{d}x^1/\mathrm{d}t = -\beta/\alpha$, which includes three important cases shown in figure 4: (*a*), two wave dislocations tangentially collide at the bifurcation point. (*b*) Two, two wave dislocations merge into one wave dislocation at the bifurcation point. (*c*), one wave dislocation splits into two wave dislocations at the bifurcation point. Thirdly, $\alpha = 0$, $\gamma \neq 0$, $\Delta = 4(\beta^2 - \alpha\gamma) > 0$, from equation (25) we have $\mathrm{d}t/\mathrm{d}x^1 = 0$ and $-2\beta/\gamma$. As shown in figure 5, there are two important cases: (*a*) One wave dislocation splits into three wave dislocations at the bifurcation point. (*b*) Three wave dislocations merge into one at the bifurcation point. Finally, $\alpha = \gamma = 0$, equations (24) and (25) give respectively $\mathrm{d}x^1/\mathrm{d}t = 0$ and $\mathrm{d}t/\mathrm{d}x^1 = 0$. This case is obvious as in figure 6, which is similar to the third situation.

Now, the branch conditions for generating, annihilating, colliding, splitting and merging of wave dislocations are given. When $D^0(\phi/x)|_{(\vec{\rho}_l,t)} = 0$ and $D^1(\phi/x)|_{(\vec{\rho}_l,t)} \neq 0$, or $D^0(\phi/x)|_{(\vec{\rho}_l,t)} = 0$ and $D^2(\phi/x)|_{(\vec{\rho}_l,t)} \neq 0$, two wave dislocations generate or annihilate (see figure 2). When $D^j(\phi/x)|_{(\vec{\rho}_l,t)} = 0$, $j = 0, 1, 2$, two wave dislocations collide (see

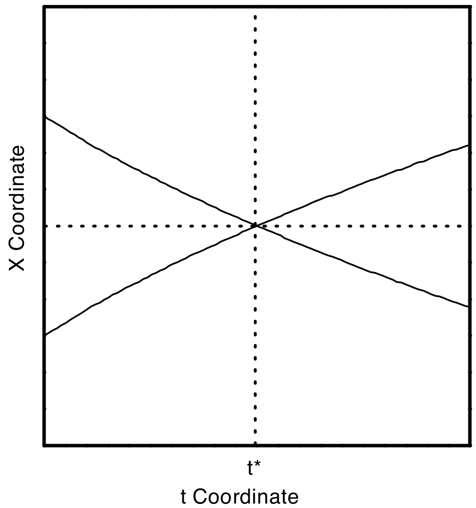

Figure 3. Two wave dislocations collide with different directions of motion at the bifurcation point.

figures 3 and 4(*a*)), one wave dislocation splits into two or three wave dislocations (see figures 4(*c*), 5(*a*) and 6(*b*)), or several wave dislocations merge into one wave dislocation (see figures 4(*b*), 5(*b*) and 6(*a*)). The identical conservation of the topological charges shows the sum of the topological charges of final wave dislocation(s) must be equal to that of the initial wave dislocation(s) at the bifurcation point.

4. Application of the branch conditions

The branch conditions obtained in the last section can be used to find a branch point and to determine which branch process will happen at the branch point. We will briefly review these branch conditions, with particular reference to the applications of the branch conditions. We give three simple examples to show how to use the branch conditions.

One example given by Nye and Berry [9] is

$$\psi(x, y, t) = \{\omega t - \mathrm{i}k^2x^2 + (A + \mathrm{i}B)k(x - \mathrm{i}y\} \exp[\mathrm{i}(kz - \omega t)] \tag{26}$$

where $x^0 = t$, $x^1 = x$ and $x^2 = y$. The positions of the wave dislocations are determined by

$$\phi^1(x, y, t) = \omega t + Akx + Bky = 0 \qquad \phi^2(x, y, t) = -k^2x^2 + Bkx - Aky = 0 \tag{27}$$

from which one can find the branch points satisfying

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi^1, \phi^2}{x, y}\right) = \det\begin{pmatrix} Ak & Bk \\ Bk - 2k^2x & -Ak \end{pmatrix} = 0 \tag{28}$$

i.e.

$$D^0\left(\frac{\phi}{x}\right) = k^2(-A^2 - B^2 + 2Bkx) = 0. \tag{29}$$

The solution of equations (27) and (29) gives a branch point:

$$x^* = \frac{A^2 + B^2}{2Bk} \qquad y^* = \frac{B^4 - A^4}{4AB^2k} \qquad t^* = \frac{-(A^2 + B^2)^2}{4AB\omega}. \tag{30}$$

Since the other two Jacobians at the branch point

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(x^*,y^*,t^*)} = D\left(\frac{\phi^1, \phi^2}{y, t}\right)\bigg|_{(x^*,y^*,t^*)} = Ak\omega \neq 0 \tag{31}$$

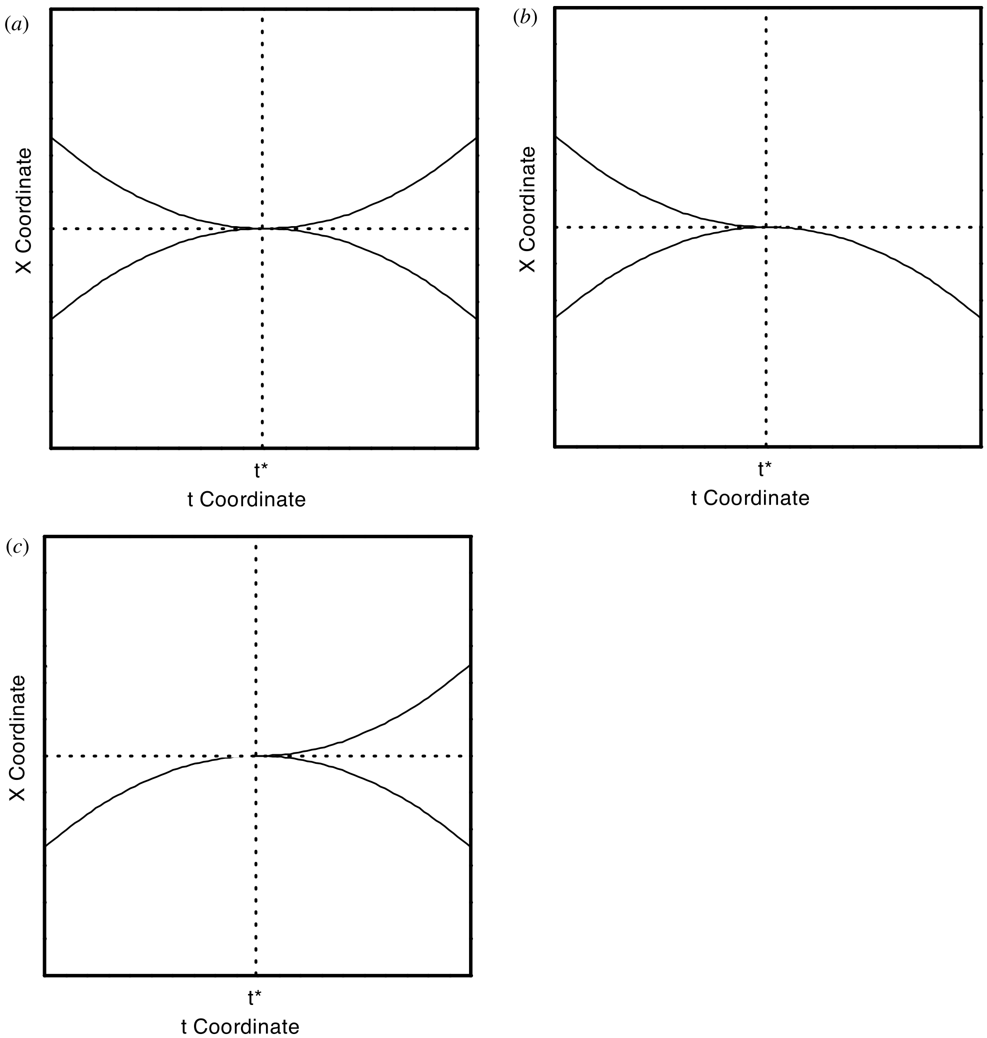

Figure 4. Wave dislocations have the same direction of motion. (*a*) Two wave dislocations tangentially collide at the bifurcation point. (*b*) Two wave dislocations merge into one wave dislocation at the bifurcation point. (*c*) One wave dislocation splits into two wave dislocations at the bifurcation point.

and

$$D^2\left(\frac{\phi}{x}\right)\bigg|_{(x^*,y^*,t^*)} = D\left(\frac{\phi^1,\phi^2}{t,x}\right)\bigg|_{(x^*,y^*,t^*)} = (B-2kx^*)k\omega \neq 0 \tag{32}$$

the branch point (x^*, z^*, t^*) is a limit point. From equations (27), one can get that

$$\frac{\mathrm{d}^2 t}{\mathrm{d}x^2}\bigg|_{(x^*,y^*,t^*)} = \frac{2Bk^2}{A\omega} \tag{33}$$

where $k^2 > 0$ and $\omega > 0$. According to equation (19), if $AB > 0$, i.e. $\mathrm{d}^2t/\mathrm{d}x^2\,|_{(x^*,z^*,t^*)} > 0$, two wave dislocations are created (see figure 2(*a*)) and if $AB < 0$, i.e. $\mathrm{d}^2t/\mathrm{d}x^2\,|_{(x^*,y^*,t^*)} < 0$, two wave dislocations annihilate (see figure 2(*b*)) at the limit point (x^*, y^*, t^*), which agree with the results obtained in [9].

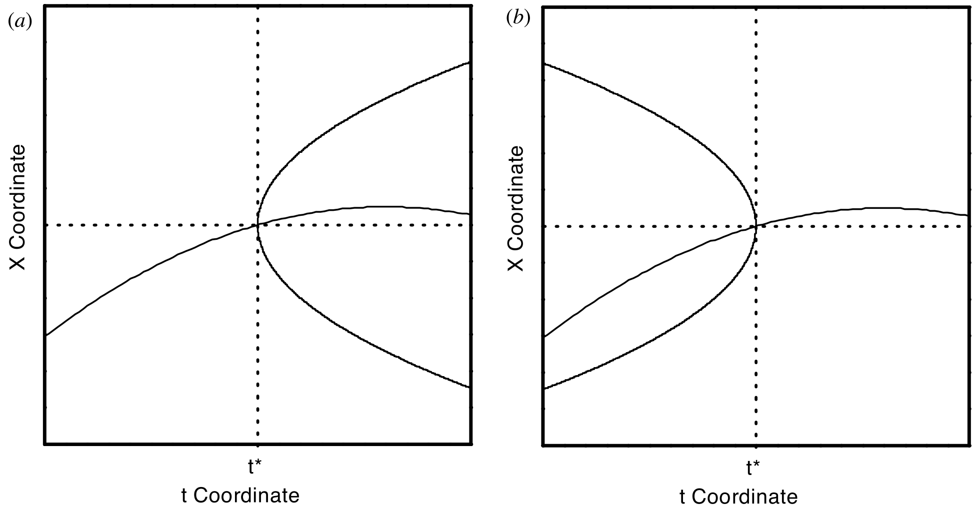

Figure 5. (*a*) One wave dislocation splits into three wave dislocations at the bifurcation point. (*b*) Three wave dislocations merge into one wave dislocation at the bifurcation point.

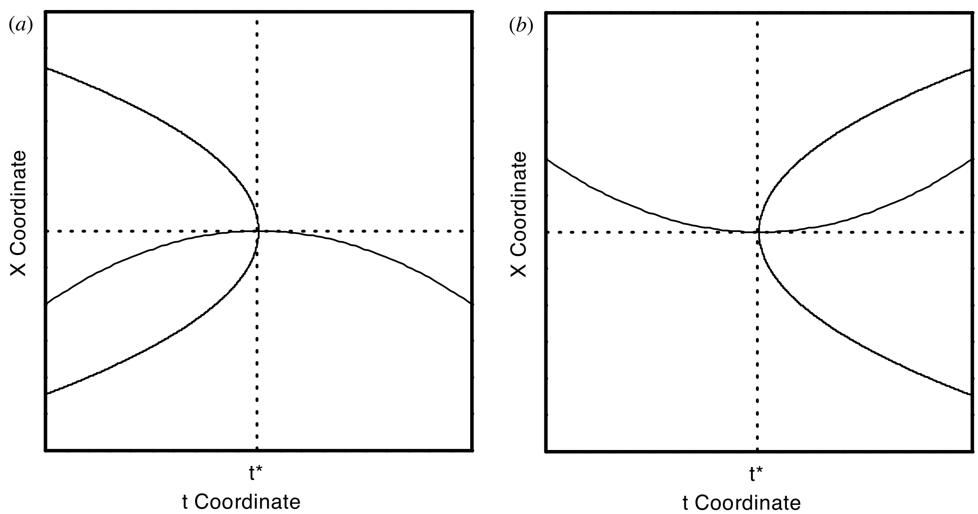

Figure 6. This case is similar to figure 5. (*a*) Three wave dislocations merge into one wave dislocation at the bifurcation point. (*b*) One wave dislocation splits into three wave dislocations at the bifurcation point.

It is also easy to find the branch point in another complex scalar wave [9]

$$\psi(x, z, t) = \{Akx + k^2x^2 + \mathrm{i}B(kz - \omega t)^2 + \mathrm{i}kz\} \exp[\mathrm{i}(kz - \omega t)] \tag{34}$$

whose wave dislocations are determined by

$$\phi^1(x, z, t) = Akx + k^2x^2 = 0 \qquad \phi^2(x, z, t) = B(kz - \omega t)^2 + kz = 0 \tag{35}$$

and whose branch points satisfy equation (35) and

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi^1, \phi^2}{x, z}\right) = \det\begin{pmatrix} Ak + 2k^2x & 0 \\ 0 & k + 2Bk(kz - \omega t) \end{pmatrix} = 0 \tag{36}$$

where $x^0 = t$, $x^1 = x$ and $x^2 = z$. We can obtain two branch points

$$t^*_{1,2} = \frac{1}{4B\omega} \qquad z^*_{1,2} = -\frac{1}{4Bk} \qquad x^*_{1,2} = 0 \qquad \text{and} \qquad -A/k \tag{37}$$

at which the other two Jacobians are

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(x^*,z^*,t^*)} = \det\begin{pmatrix} 0 & 0 \\ k + 2Bk(kz^* - \omega t^*) & -2B\omega(kz^* - \omega t^*) \end{pmatrix} = 0 \tag{38}$$

and

$$D^2\left(\frac{\phi}{x}\right)\bigg|_{(x^*,z^*,t^*)} = \det\begin{pmatrix} 0 & Ak + 2k^2x^* \\ -2B\omega(kz^* - \omega t^*) & 0 \end{pmatrix} \neq 0. \tag{39}$$

So, (x^*_1, z^*_1, t^*_1) and (x^*_2, z^*_2, t^*_2) are two limit points, near which two wave dislocations generate or annihilate.

From the above examples, the branch conditions appear to be complicated compared with [9] in that these examples are simpler, but the actual situations on wave dislocations in experiments may be much more complicated than these examples.

The limit points are subject to the branch conditions, and so are the bifurcation points. There is an example bearing a bifurcation point given by Nye and Berry [9]:

$$\psi(x, z, t) = \{\tfrac{1}{3}k^3x^3 + B(kz - \omega t) + \mathrm{i}k^2xz\}\exp[\mathrm{i}(kz - \omega t)]. \tag{40}$$

Wave dislocations are located at the space-time point satisfying

$$\phi^1(x, z, t) = \tfrac{1}{3}k^3x^3 + B(kz - \omega t) = 0 \qquad \phi^2(x, z, t) = k^2xz = 0. \tag{41}$$

Branch point should satisfy not only equation (41) but also

$$D^0\left(\frac{\phi}{x}\right) = D\left(\frac{\phi^1, \phi^2}{x, z}\right) = \det\begin{pmatrix} k^3x^2 & Bk \\ k^2z & k^2x \end{pmatrix} = 0 \tag{42}$$

that is,

$$k^5x^3 - Bk^3z = 0 \tag{43}$$

thus, we obtain a branch point

$$t^* = 0 \qquad z^* = 0 \qquad x^* = 0 \tag{44}$$

at which the values of $D^1(\phi/x)$ and $D^2(\phi/x)$ are calculated as follows:

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(x^*,z^*,t^*)} = \det\begin{pmatrix} Bk & -B\omega \\ k^2x^* & 0 \end{pmatrix} = Bk^2\omega x^* = 0 \tag{45}$$

$$D^2\left(\frac{\phi}{x}\right)\bigg|_{(x^*,z^*,t^*)} = \det\begin{pmatrix} -B\omega & k^3x^{*2} \\ 0 & k^2z^* \end{pmatrix} = -Bk^2\omega z^* = 0. \tag{46}$$

Since $D^1(\phi/x)$ and $D^2(\phi/x)$ both are equal to zero, this point must be a bifurcation point according to our theory. In the neighbourhood of the bifurcation point two wave dislocations collide at (x^*, z^*, t^*) (see figure 3).

From these three examples we see how to use the branch conditions. One can apply them into both numerical and experimental branch processes of wave dislocations. These conditions will help those experts in optics to find a branch point and the concrete branch process in the neighbourhood of it, and to give a deep insight into wave dislocations.

5. Conclusions

First, the densities of wave dislocations are obtained directly from the definition of topological charges of wave dislocations (winding numbers of zero points of the two-dimensional vector field, i.e. complex scalar wave in light beams). Secondly, the branch conditions for generating, annihilating, colliding, splitting and merging of wave dislocations are obtained. When $D^0(\phi/x)|_{(\vec{\rho}_l,t)} = 0$ and $D^1(\phi/x)|_{(\vec{\rho}_l,t)} \neq 0$, or $D^0(\phi/x)|_{(\vec{\rho}_l,t)} = 0$ and $D^2(\phi/x)|_{(\vec{\rho}_l,t)} \neq 0$, two wave dislocations generate or annihilate. When $D^j(\phi/x)|_{(\vec{\rho}_l,t)} = 0$, $j = 0, 1, 2$, the wave dislocations collide, split or merge. Thirdly, we find the result that the velocity of wave dislocations is infinite when they are annihilating or generating, which is obtained only from the topological properties of the complex scalar waves. Last, we apply the branch conditions in some simple examples in the literature and hope they have widespread applications in the future.

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